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Journal of Geometry and Physics 32 (1999) 131–159

JOURNAL OF GEOMETRY AND PHYSICS

A Dirichlet criterion for the stability of periodic and relative periodic orbits in Hamiltonian systems

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Received 1 October 1998; received in revised form 29 March 1999

Abstract

The classical Dirichlet criterion on the stability of equilibria in Hamiltonian systems is generalized to the orbital stability of periodic orbits. The result obtained is adapted to the study of the stability of relative periodic orbits in Hamiltonian systems with symmetry that lie at regular level sets of the corresponding momentum map. © 1999 Elsevier Science B.V. All right reserved.

Subj. Class.: Dynamical systems
1991 MSC: 34D35; 58F05
Keywords: Dirichlet criterion; Stability; Periodic and relative periodic orbits; Hamiltonian systems

1. Introduction

Let $X \in \mathfrak{X}(M)$ be a dynamical system on the manifold M, and let $m \in M$ be an equilibrium that is, $X(m) = 0$. We say that the equilibrium m is *Lyapunov stable* or *nonlinearly stable* when for any open neighborhood U of m in M, there is another open neighborhood $V \subset U$ of m such that if F_t is the flow associated to X, then $F_t(z) \in U$, for any $z \in V \subset U$ and for all $t > 0$.

A very important problem in the theory of dynamical systems consists of determining the stability of a given equilibrium. In this direction, the simplest and best known stability criterion is the following.

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0393-0440/99/$ – see front matter
PII: S0393-0440(99)00025-X

Theorem 1.1 (Dirichlet). *Let $X \in \mathfrak{X}(M)$ be a dynamical system on the manifold M, and let $m \in M$ be an equilibrium. Let $C \in C^{\infty}(M)$ be a conserved quantity of X that is, $C \circ F_t = C$ for all time t, where F_t is the flow associated to X. If C is such that $\mathbf{d}C(m) = 0$ and the quadratic form $\mathbf{d}^2C(m)$ is definite, then the equilibrium m is stable.*

This sufficient condition for stability has special relevance in the Hamiltonian case that is, when M is a symplectic manifold with closed nondegenerate two-form ω and X is a Hamiltonian vector field with corresponding Hamiltonian function $h \in C^{\infty}(M)$. In this case, part of the hypotheses of Dirichlet's theorem are automatically satisfied. In particular, the Hamiltonian h is always a conserved quantity (we focus exclusively on autonomous systems) such that, by the nondegeneracy of the symplectic form, $\mathbf{d}h(m) = 0$. However, in very important situations $\mathbf{d}^2h(m)$ is not definite so one has to either use different conserved quantities or look at other stability criteria like the *KAM theorem* (see [5,3,6]). Regarding the first solution, there are unfortunately only a few examples in which the integrals needed are available; moreover, some genericity results of Robinson [46], together with the closing lemma of Pugh [44,45] prove that most Hamiltonian vector fields do not have integrals of motion other than the Hamiltonian function. As to the KAM theorem, it has been instrumental in the proof of the orbital stability of very relevant physical systems. For example, in the restricted three-body problem, this theorem is able to prove the stability of the Trojans and predicts the gaps in the distribution of asteroids between Mars and Jupiter (see [49,21]). Over the years the KAM theorem has been extended to a very wide class of systems. However, the persistence of the tori that it predicts guarantees stability only in the case in which the dimension of the configuration space equals two; in higher dimensions, phenomena like the Arnold diffusion may spoil the stability. All in all, the stability in the sense of the definition stated in the first paragraph of this section remains an open problem, specially when the dimension of the space is bigger than two.

Nevertheless, in the way to understanding this problem, Dirichlet's theorem has been a key point of reference, mainly due to its dimension independent nature and has given rise to an approach to the theory of stability usually known as *energetics*. In particular, Moser [38] has formulated the concept of *almost stability* which amounts to the existence of an approximate conserved quantity, given in the form of a real formal power series (possibly divergent), that satisfies Dirichlet's theorem. Moser carried out some estimates [37] which showed that, in the presence of almost stability, solutions starting sufficiently close to the almost stable equilibrium m will remain in a given neighborhood of m for extremely long times, which is more than enough for applications in physics. In the same spirit of *finite time stability* or *effective stability* we also have the results of Nekhoroshev [39] and all its subsequent improvements and implementations (see [14,19,8–10,15] and references therein).

Dirichlet's theorem has been adapted to the study of the stability of relative equilibria in symmetric Hamiltonian systems. In this situation it receives the names of *Arnold* [4], *energy-Casimir* [20], or *energy–momentum method* [47,25,43,40,36,24,42]. All these techniques are based on the definiteness of the second variation of an augmented energy function (which in infinite dimensions needs to be replaced with certain convexity estimates) and, as in the case of Dirichlet's theorem, they provide only sufficient stability conditions.

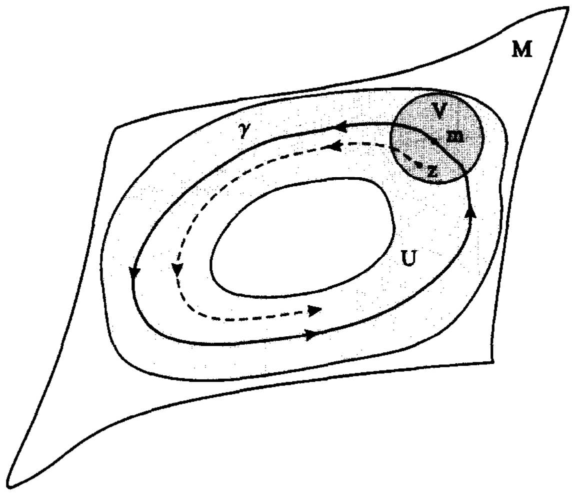

Fig. 1. Birkhoff's orbital stability.

In this paper we will formulate a Dirichlet-like sufficient condition for the stability of periodic orbits (Theorem 3.1) and relative periodic orbits (Theorem 5.4) in the Hamiltonian context. For terminological convenience and to differentiate these results from the situations dealing with equilibria and relative equilibria, we will call our conditions the *energy-integrals* and the *symmetric energy-integrals* methods, respectively. To fix ideas, the notion of stability (Fig. 1) we will be interested in is expressed in the following definition, introduced by Birkhoff.

Definition 1.2. Let $X \in \mathfrak{X}(M)$ be a dynamical system on the manifold M, and let γ be a periodic orbit of (M, X) such that $m \in \gamma$. We say that γ is *orbitally stable*, or that m is a *stable periodic point*, if for any open neighborhood U of γ in M, there is an open neighborhood V of m such that if F_t is the flow associated to X, then $F_t(z) \in U$, for any $z \in V \subset U$ and for all $t > 0$.

Let us emphasize that the similarity of the results obtained with Dirichlet's theorem makes them share with it his advantages, as the ease of formulation and use in particular cases, but also his limited range of applicability. Nevertheless, as Dirichlet's result is the first construction block in other kinds of stability criteria in the framework of equilibria already mentioned, so should be the case for the energy and symmetric energy-integrals method for the case of periodic and relative periodic orbits. This will be dealt with in a future work.

The proof strategy of our results follows the remarkable technique introduced by Patrick [43], based on the use of certain "penalty functions". In a first step, we will restrict ourselves to relative periodic orbits for which the value of the momentum map is a regular value. The singular case will be the subject of a future paper. Also, in this first approach, only the finite-dimensional case will be treated. However, it is expected that

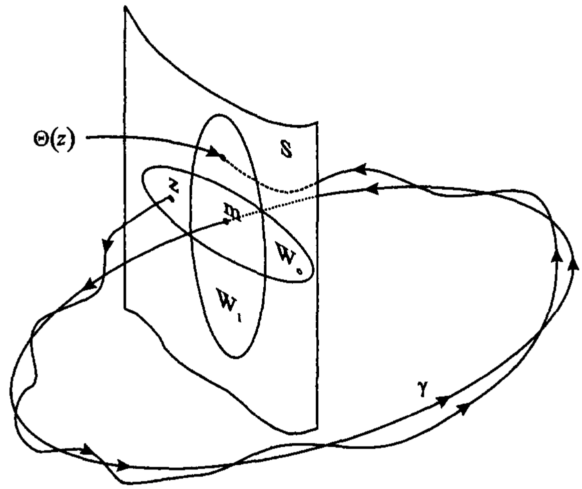

Fig. 2. Poincaré section and Poincaré map.

infinite-dimensional systems can also be treated in this fashion, at least at "formal stability" level (see [20]).

The structure of the paper is as follows. Section 2 recalls standard notation and results; we will use the terminology and conventions in [1]. Section 3 presents the energy-integrals method, proves the main theorem and shows its application to some elementary examples. Section 5 extends the energy-integrals method to study the stability of relative periodic orbits. For the convenience of the reader, all the prerequisites on reduction and normal forms are briefly reviewed in Section 4.

2. Preliminary concepts

Two important tools in the study of periodic orbits are the local transversal sections and the Poincaré maps (Fig. 2). We briefly review these concepts and their principal properties (see [1]).

Definition 2.1. Let $X \in \mathfrak{X}(M)$ be a vector field on the manifold M. A *local transversal section* of X at $m \in M$ is a submanifold $S \subset M$ of codimension one with $m \in S$ and such that for all $s \in S$, $X(s)$ is not contained in T_sS. Therefore $T_sM = T_sS \oplus \operatorname{span}\{X(s)\}$.

Let $F : \mathcal{D}_X \subset M \times \mathbb{R} \to M$ be the flow of X which includes a closed orbit γ through m with period τ; $\mathcal{D}_X$ is the domain of the flow, an open subset of $M \times \mathbb{R}$. A *Poincaré map* of γ is a mapping $\Theta : W_0 \to W_1$ where:

(PM1) $W_0, W_1 \subset S$ are open neighborhoods of $m \in S$, and Θ is a diffeomorphism;
(PM2) there is a continuous function $\delta : W_0 \to \mathbb{R}$, called the *period shift*, such that $(s, \tau - \delta(s)) \in \mathcal{D}_X$, and $\Theta(s) = F(s, \tau - \delta(s))$ for all $s \in W_0$;
(PM3) if $t \in (0, \tau - \delta(s))$, then $F(s, t) \notin W_0$.

A fundamental theorem guarantees the existence and uniqueness of Poincaré maps for closed orbits in arbitrary dynamical systems. By uniqueness we mean that if $\Theta' : W_0' \to W_1'$ is another Poincaré map constructed using the section S' through $m' \in \gamma$, then Θ and Θ' are locally conjugate, that is, there are open neighborhoods W_2 of $m \in S$, W_2' of $m' \in S'$, and a diffeomorphism $\mathcal{H} : W_2 \to W_2'$, such that $W_2 \subset W_0 \cap W_1$, $W_2' \subset W_0' \cap W_1'$, and the diagram

$$\begin{array}{ccc} \Theta^{-1}(W_2) \cap W_2 & \xrightarrow{\Theta} & W_2 \cap \Theta(W_2) \\ \mathcal{H}\downarrow & & \downarrow\mathcal{H} \\ W_2' & \xrightarrow{\Theta'} & S' \end{array}$$

commutes.

If the manifold M is symplectic, with symplectic form ω, and the vector field X is a Hamiltonian dynamical system associated to the function $h \in C^\infty(M)$ (we will denote X by X_h in this case) then these additional structures allow us to choose the elements of Definition 2.1 with the properties stated in the following theorem (which is Proposition 8.1.3 in [1]). Note that if γ is a closed orbit of X_h, then we may assume that γ lies in a regular energy surface Σ_e of h since near γ, $\mathbf{d}h$ must be nonzero.

Theorem 2.2. *Let (M, ω) be a symplectic manifold, $h \in C^\infty(M)$, and γ a closed orbit of X_h lying in the regular energy surface Σ_e. Then, there exists a local transversal section S at $m \in \gamma$ and a Poincaré map $\Theta : W_0 \to W_1$ on S, such that the following hold:*

(i) *(W_0, ω_0) and (W_1, ω_1) are contact manifolds where $\omega_j = i_j^*\omega$, $i_j : W_j \hookrightarrow M$ being the natural inclusion and $j \in \{0, 1\}$;*

(ii) *Θ is a canonical transformation; that is, Θ preserves h, and there is a function $\delta \in C^\infty(W_0)$ such that $\Theta^*\omega_1 = \omega_0 - \mathbf{d}\delta \wedge \mathbf{d}h$; moreover, δ is the period shift of the Poincaré map described in Definition* 2.1;

(iii) *there exist $\epsilon > 0$ and regular energy surfaces $\Sigma_{e'}$ for $e' \in (e - \epsilon, e + \epsilon)$, such that $(S_{e'} := S \cap \Sigma_{e'}, \omega_{e'})$ is a symplectic submanifold of M of codimension two and $\Theta|_{W_0 \cap S_{e'}}$ is a symplectomorphism onto $W_1 \cap S_{e'}$, where $\omega_{e'} = i^*\omega$ and $i : S_{e'} \hookrightarrow M$ is the natural inclusion.*

Another concept that will be used ubiquitously is the *Hessian*, whose definition and properties we recall in what follows. We will use here the definition of the Hessian from differential topology (see [35]). If M is a smooth manifold and $f \in C^\infty(M)$, suppose that $m \in M$ is a *critical point* of f, that is, $\mathbf{d}f(m) = 0$.

Definition 2.3. The *Hessian* of $f \in C^\infty(M)$ at the critical point $m \in M$ is the symmetric bilinear form $\mathbf{d}^2 f(m) : T_m M \times T_m M \to \mathbb{R}$, given by

$$\mathbf{d}^2 f(m)(v, w) := v[W[f]],$$

where $v, w \in T_m M$ and $W \in \mathfrak{X}(M)$ is an arbitrary extension of $w \in T_m M$ to a vector field on M, that is, $W(m) = w$.

Remark 2.4. *The requirement that $m \in M$ is a critical point of f is crucial and guarantees the correctness of Definition 2.3, that is, the value of $\mathbf{d}^2 f(m)(v, w)$ does not depend on the extension W of w. In addition, m being a critical point of f, allows one to easily prove the symmetry of $\mathbf{d}^2 f(m)$. The proof of the following proposition follows directly from the definitions.*

Proposition 2.5. *Let $m \in M$ and $n \in N$, with M and N being smooth differentiable manifolds. Let $\psi : M \to N$ be a smooth map such that $\psi(m) = n$ and let $f \in C^\infty(N)$ with $\mathbf{d}f(n) = 0$. Then $\mathbf{d}^2(\psi^* f)(m) = T_m^*\psi(\mathbf{d}^2 f(n))$, that is, for any $v, w \in T_m M$:*

$$\mathbf{d}^2(\psi^* f)(m)(v, w) = \mathbf{d}^2 f(n)(T_m\psi \cdot v, T_m\psi \cdot w).$$

In particular, if S is a submanifold of M, $f \in C^\infty(M)$, and $m \in S$ then,

$$\mathbf{d}^2 f(m)|_{T_m S \times T_m S} = \mathbf{d}^2(f|_S)(m).$$

Finally, the proof of our first main theorem will require the use of a lemma due to Patrick (see [43] for a proof).

Lemma 2.6. *Let A and B be bilinear forms on a finite-dimensional vector space. Suppose that A is positive semidefinite and that B is positive definite on* ker A. *Then, there exists $r > 0$ such that $A + \epsilon B$ is positive definite for all $\epsilon \in (0, r)$.*

3. Orbital stability and the energy-integrals method

We shall work generally on a Poisson manifold, that is, a manifold M whose space of smooth functions $C^\infty(M)$ admits a bracket $\{\cdot\,,\cdot\}$ relative to which it is a Lie algebra and the Leibniz identity holds in each argument. The Hamiltonian vector field X_h given by a function $h \in C^\infty(M)$ is defined (as a derivation) by the relation $X_h = \{\cdot\,, h\}$. The elements of the center of the Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$ are called *Casimir functions*. The triplet $(M, \{\cdot\,,\cdot\}, h)$ is called a *Poisson system*. Any Poisson manifold is partitioned into symplectic leaves, which are connected immersed symplectic manifolds of M inducing the Poisson bracket of M. The tangent space at m to a leaf consists of all vectors that are equal to the value of some Hamiltonian vector field at m. The symplectic leaves are invariant under the flow of any Hamiltonian vector field.

Theorem 3.1 (The energy-integrals method). *Let γ be a periodic orbit of the Poisson system $(M, \{\cdot, \cdot\}, h)$ through the point $m \in M$. Let $C_1, C_2, \ldots, C_n \in C^\infty(M)$ be a set of conserved quantities (integrals of the motion) for which*

$$\mathbf{d}(C_1 + \cdots + C_n)(m) = 0.$$

If the quadratic form

$$\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W \times W}$$

is definite for some (and hence for any) subspace $W \subset T_m M$ such that

$$\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) = W \oplus \operatorname{span}\{\gamma'(m)\},$$

then γ is orbitally stable. If $W = \{0\}$ (in particular, if $\dim M = 2$), then γ is always orbitally stable.

Proof. We first prove the case $W \neq \{0\}$ and we begin by showing that the result does not depend on the choices of $m \in \gamma$ and W. Indeed, if $\mathbf{d}(C_1 + \cdots + C_n)(m) = 0$ and F_t is the flow of the Hamiltonian vector field X_h, then for any $t > 0$ and any $v, w \in T_m M$ we have

$$\begin{aligned}
&\mathbf{d}(C_1 + \cdots + C_n)(F_t(m))(T_m F_t(v), T_m F_t(w)) \\
&\quad = {F_t}^*(\mathbf{d}(C_1 + \cdots + C_n)(m))(v, w) \\
&\quad = \mathbf{d}({F_t}^*(C_1 + \cdots + C_n))(m)(v, w) \\
&\quad = \mathbf{d}(C_1 + \cdots + C_n)(m)(v, w),
\end{aligned}$$

since ${F_t}^* \circ \mathbf{d} = \mathbf{d} \circ {F_t}^*$ and $C_1, C_2, \ldots, C_n$ are invariant under F_t. If W is a complement to $\operatorname{span}\{\gamma'(m)\}$ in $\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$, then for any $t > 0$, $T_m F_t(W)$ is a complement to $\operatorname{span}\{\gamma'(F_t(m))\}$ in $\ker \mathbf{d}C_1(F_t(m)) \cap \cdots \cap \ker \mathbf{d}C_n(F_t(m))$. Moreover, $\mathbf{d}^2(C_1 + \cdots + C_n)(m)|_{W \times W}$ is definite iff $\mathbf{d}^2(C_1 + \cdots + C_n)(F_t(m))|_{T_m F_t \cdot W \times T_m F_t \cdot W}$ is definite, since Proposition 2.5 and conservation of $C_1, \ldots, C_n$, imply for any $v, w \in T_m M$:

$$\begin{aligned}
&\mathbf{d}^2(C_1 + \cdots + C_n)(F_t(m))(T_m F_t(v), T_m F_t(w)) \\
&\quad = \mathbf{d}^2({F_t}^* C_1 + \cdots + {F_t}^* C_n)(m)(v, w) \\
&\quad = \mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w).
\end{aligned}$$

The statement of the theorem does therefore not depend on the choice of the point $m \in \gamma$.

The choice of W is also irrelevant since $\mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w) = 0$ whenever $v \in \operatorname{span}\{\gamma'(m)\}$. Indeed, since we can assume without loss of generality that $v = X_h(m)$, the definition of the Hessian implies

$$\begin{aligned}
&\mathbf{d}^2(C_1 + \cdots + C_n)(m)(v, w) = w[X_h(C_1 + \cdots + C_n)] \\
&\quad = w[\{C_1, h\} + \cdots + \{C_n, h\}] = 0.
\end{aligned}$$

Thus, the statement of the theorem does not depend on the choices of $m \in \gamma$ and $W \subset T_m M$ as long as $\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) = W \oplus \operatorname{span}\{\gamma'(m)\}$.

We proceed to the proof of the theorem by defining for a fixed $m \in \gamma$

$$\begin{aligned}
f_1 &= C_1 - C_1(m) + \cdots + C_n - C_n(m), \\
f_2 &= (C_1 - C_1(m))^2 + \cdots + (C_n - C_n(m))^2.
\end{aligned}$$

The hypothesis of the theorem clearly implies that

$$\mathbf{d}f_1(m) = \mathbf{d}f_2(m) = 0.$$

Let S be a local transversal section to γ at $m \in \gamma$. Let Z be the subspace

$$Z := T_m S \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m).$$

By the properties of the local transversal section we have $\gamma'(m) \notin T_m S$ and hence

$$Z \cap \operatorname{span}\{\gamma'(m)\} = \{0\}.$$

Since $\operatorname{span}\{\gamma'(m)\} \subset \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$ and $T_m M = T_m S \oplus \operatorname{span}\{\gamma'(\mathrm{m})\}$ we get

$$\begin{aligned}
\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) &= T_m M \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)\\
&= (T_m S \oplus \operatorname{span}\{\gamma'(m)\}) \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)\\
&= (T_m S \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)) \oplus \operatorname{span}\{\gamma'(m)\}\\
&= Z \oplus \operatorname{span}\{\gamma'(m)\},
\end{aligned}$$

that is, Z is a complement to $\operatorname{span}\{\gamma'(m)\}$ in $\ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$. Since f_1 and $C_1 + \cdots + C_n$ differ by a constant, the hypothesis of the theorem implies that the form $\mathbf{d}^2 f_1(m)|_{Z\times Z}$ is definite. Using Proposition 2.5 we have

$$\mathbf{d}^2 f_1(m)|_{Z\times Z} = (\mathbf{d}^2 f_1(m)|_{T_m S\times T_m S})|_{Z\times Z} = \mathbf{d}^2(f_1|_S)(m)|_{Z\times Z}.$$

Hence the hypothesis of the theorem is equivalent to saying that $\mathbf{d}^2(f_1|_S)(m)|_{Z\times Z}$ is definite.

We prove now that Z is the kernel of $\mathbf{d}^2(f_2|_S)(m)$. Let $v_1, v_2 \in T_m S$, such that $v_i = \mathrm{d}/\mathrm{d}t|_{t=0} c_i(t)$, with $c_i(t) \in S$ for any t and $i \in \{1, 2\}$. Let $X_{v_i} \in \mathfrak{X}(S)$ be an extension of v_i to a vector field on S whose flow is denoted by $F_t^{v_i}$. Then, by definition

$$\begin{aligned}
\mathbf{d}^2(f_2|_S)(m)(v_1, v_2) &= v_1[X_{v_2}[f_2]]\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f_2(F_s^{v_2}(c_1(t)))\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (C_1(F_s^{v_2}(c_1(t))) - C_1(m))^2 + \cdots + (C_n(F_s^{v_2}(c_1(t))) - C_n(m))^2\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} 2(C_1(F_s^{v_2}(m)) - C_1(m))\mathbf{d}C_1(F_s^{v_2}(m)) \cdot T_m F_s^{v_2}(v_1)\\
&\quad + \cdots + \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} 2(C_n(F_s^{v_2}(m)) - C_n(m))\mathbf{d}C_n(F_s^{v_2}(m)) \cdot T_m F_s^{v_2}(v_1)\\
&= 2[(\mathbf{d}C_1(m) \cdot v_2)(\mathbf{d}C_1(m) \cdot v_1) + \cdots + (\mathbf{d}C_n(m) \cdot v_2)(\mathbf{d}C_n(m) \cdot v_1)].
\end{aligned}$$

Hence $v_1 \in \ker \mathbf{d}^2(f_2|_S)(m)$ iff for any $v_2 \in T_m S$, we have

$$(\mathbf{d}C_1(m) \cdot v_1)(\mathbf{d}C_1(m) \cdot v_2) + \cdots + (\mathbf{d}C_n(m) \cdot v_1)(\mathbf{d}C_n(m) \cdot v_2) = 0.$$

Taking in this relation $v_2 = v_1$, this implies that $\mathbf{d}C_1(m) \cdot v_1 = \cdots = \mathbf{d}C_n(m) \cdot v_1 = 0$ and hence $v_1 \in \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m) \cap T_m S = Z$. Conversely, if $v_1 \in Z = T_m S \cap \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m)$ the above relation is satisfied trivially for all $v_1 \in T_m S$. Therefore,

$$Z = \ker \mathbf{d}^2(f_2|_S)(m).$$

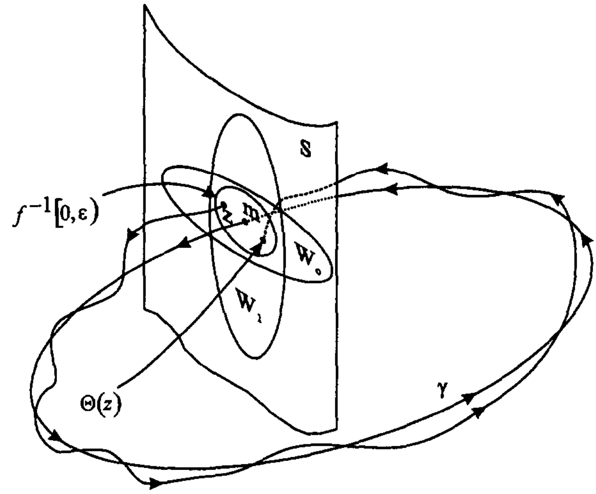

Fig. 3. Construction of A.

Using all these remarks and the obvious positive semidefiniteness of $\mathbf{d}^2(f_2|_S)(m)$, Lemma 2.6 guarantees the existence of some $a > 0$ for which the function f defined by

$$f := af_1 + f_2 \tag{3.1}$$

is such that $\mathbf{d}^2(f|_S)(m)$ is positive definite.

Let now V be an open neighborhood of γ. With the notation of Definition 2.1, the Morse Lemma allows us to choose S and $\epsilon > 0$ such that $f \geqslant 0$ on S and

$$f^{-1}[0,\epsilon) \cap S \subseteq V \cap W_0 \cap W_1.$$

Notice that since f is a conserved quantity, if $z \in W_0 \cap W_1 \cap f^{-1}[0,\epsilon)$ then $F_{\tau-\delta(z)}(z) \in W_0 \cap W_1 \cap f^{-1}[0,\epsilon)$ (see Fig. 3). Let $D_V = \inf\{d(x,\gamma) | x \in \overline{V} \setminus V\}$, where d is the distance function on M associated to some Riemannian metric on M (we assume that M is paracompact and hence there is always some Riemannian metric on it). The compactness of γ and the openness of V guarantee that D_V is never 0.

If $A = W_0 \cap W_1 \cap f^{-1}[0,\epsilon)$, we define the map:

$$\begin{aligned} D : A &\longrightarrow \mathbb{R} \\ z &\longmapsto D(z) := \max_{t\in[0,\tau-\delta(z)]} d(F_t(z),\gamma). \end{aligned}$$

Note that $D(m) = 0$. By the continuity of D, we can choose $\epsilon > 0$ (and therefore A) small enough so that $D(z) < D_V/2$, for any $z \in A$. Define the open neighborhood U of γ by

$$U := \{F_t(z') | z' \in A, t \geq 0\}.$$

We shall prove below that $F_t(U) \subset V$ for all $t \geq 0$. In order to see this, note that, by construction, U is invariant under the flow F_t and hence the claim is proved if we show that

$U \subset V$. Let us suppose the contrary, namely that there is an element $F_t(z') \in U$, $z' \in A$, such that $F_t(z') \notin V$. Without loss of generality we can assume that $t \in [0, \tau - \delta(z')]$ which then implies that $d(F_t(z'), \gamma) \leqslant D(z') < D_V/2$. However, since we assume that $F_t(z') \notin V$, it follows that $d(F_t(z'), \gamma) \geqslant D_V$, by the definition of D_V.

In the case $W = \{0\}$, $Z = \ker \mathbf{d}^2(f_2|_S)(m) = \{0\}$ and, therefore $\mathbf{d}^2(f_2|_S)(m)$ is positive definite, hence we do not need to apply Lemma 2.6 and, the rest of the proof follows just by taking f_2 instead of f. □

Remark 3.2. *The method is called energy-integrals since one can always use the Hamiltonian of the system, that is, its total energy, as one of the conserved quantities C_i in Theorem* 3.1.

Remark 3.3. *The requirement on the conserved quantities in the statement of the theorem about belonging to $C^\infty(M)$, that is, being defined on the entire manifold M, can be relaxed to being defined in an open neighborhood of the periodic orbit γ.*

Example 3.4. The algorithm provided by the energy-integrals method allows us to show, in a computationally straightforward manner, the orbital stability of the periodic orbits of some classical systems: the rigid body, the resonant harmonic oscillator, and the closed Keplerian orbits. Notice that the orbital stability of these motions is known, given that all these systems are integrable and all their bounded motions are periodic which, looking at the system in action–angle coordinates gives us orbital stability. In what follows we give an indication of the conserved quantities that should be used in the application of the energy-integrals method, for each particular system:

(i) *The rigid body:* the total energy and the total angular momentum.
(ii) *The resonant harmonic oscillator:* the energies of each oscillator and the generalized angular momentum.
(iii) *Closed Keplerian orbits:* the total energy, one component of the angular momentum, and one component of the Laplace–Runge–Lenz vector.

4. Systems with symmetries, reduction and normal forms

We will dedicate Section 5 to study the stability of relative periodic orbits. The analysis of these elements requires some background in reduction theory and normal forms that we will briefly review in this section. The expert may proceed directly to Section 5.

Let $(M, \omega, G, \mathbf{J}: M \to \mathfrak{g}^*, h: M \to \mathbb{R})$ be a Hamiltonian dynamical system with a symmetry given by the Lie group G acting properly on M. The symbol $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$, the Lie algebra of G. The Hamiltonian $h \in C^\infty(M)$ is G-invariant and the momentum map $\mathbf{J}$ is assumed to be equivariant. For any $\xi \in \mathfrak{g}$, we will denote by $\mathbf{J}^\xi \in C^\infty(M)$ the function defined by $\mathbf{J}^\xi(z) := \langle \mathbf{J}(z), \xi \rangle$, where $\langle \cdot , \cdot \rangle$ is the natural pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. If $m \in M$ is such that $\mathbf{J}(m) = \mu$ is a regular value of $\mathbf{J}$ whose coadjoint isotropy subgroup G_μ acts freely on the manifold $\mathbf{J}^{-1}(\mu)$, it is well known [34] that the space $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic manifold and that the dynamics induced by h reduces naturally to Hamiltonian dynamics on $\mathbf{J}^{-1}(\mu)/G_\mu$. More specifically we have the following.

Theorem 4.1 (Symplectic reduction). *Let $(M, \omega, G, \mathbf{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$ be a Hamiltonian dynamical system with a symmetry given by the left action of the Lie group G. We assume that the associated momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ is equivariant and that the Hamiltonian h is G-invariant. We also suppose that if $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$, the isotropy subgroup G_μ acts freely and properly on $\mathbf{J}^{-1}(\mu)$. Then,*

(i) *the reduced space, $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$, is endowed with the unique quotient differentiable structure for which the canonical projection*

$$\pi_\mu : \mathbf{J}^{-1}(\mu) \longrightarrow \mathbf{J}^{-1}(\mu)/G_\mu,$$

is a surjective submersion;

(ii) *the reduced space M_μ has a unique symplectic form ω_μ, characterized by*

$$\pi_\mu{}^*\omega_\mu = i_\mu{}^*\omega,$$

where $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$ is the natural inclusion;

(iii) *the Hamiltonian flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the G_μ-action, so it induces a flow F_t^μ on M_μ that is characterized by*

$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu;$$

(iv) *the flow F_t^μ is Hamiltonian on M_μ with reduced Hamiltonian function $h_\mu : M_\mu \to \mathbb{R}$ defined by*

$$h_\mu \circ \pi_\mu = h \circ i_\mu;$$

the vector fields X_h and X_{h_μ} are π_μ-related;

(v) *if $k : M \to \mathbb{R}$ is another G-invariant function, then $\{h, k\}$ is also G-invariant and*

$$\{h, k\}_\mu = \{h_\mu, k_\mu\}_{M_\mu}$$

where $\{\cdot, \cdot\}_{M_\mu}$ denotes the Poisson bracket induced by the symplectic structure on M_μ.

In what follows we will not just assume that the isotropy subgroup G_μ acts freely and properly on $\mathbf{J}^{-1}(\mu)$, but that the whole group G acts freely and properly on M. In this situation, the orbit space M/G is a smooth manifold (see [1], Theorem 4.1.20; [2], Proposition 3.5.21). Moreover, the Poisson structure on M induced by its symplectic form drops to M/G ([33], Theorem 10.7.1).

Theorem 4.2. *Under the hypotheses mentioned above, there is a unique Poisson structure $\{\cdot, \cdot\}_{M/G}$ on the quotient M/G such that the canonical projection $\pi : M \to M/G$ is a Poisson map. Moreover, $\{\cdot, \cdot\}_{M/G}$ is uniquely determined by the relation*

$$\{f, g\}_{M/G} \circ \pi = \{f \circ \pi, k \circ \pi\},$$

where $f, k : M/G \to \mathbb{R}$ are two arbitrary functions, and $\{\cdot, \cdot\}$ denotes the Poisson structure associated to the symplectic form ω in M.

Another tool that will be used in the proof of the main result in Section 5 is the *Marle–Guillemin–Sternberg (MGS) normal form*. This normal form was introduced by Marle [32] and by Guillemin and Sternberg [17,18].

The MGS normal form is a Slice Theorem in the category of Hamiltonian actions and gives a G-invariant local model around each point of (M, ω) considered as a Hamiltonian G-space. Let $m \in M$, $\mu := \mathbf{J}(m) \in \mathfrak{g}^*$, and assume, as in Theorem 4.1, that the coadjoint isotropy subgroup G_μ is compact. The vector space $V := T_m(G \cdot m)^\omega / [T_m(G \cdot m)^\omega \cap T_m(G \cdot m)] = T_m(G \cdot m)^\omega / T_m(G_\mu \cdot m)$ is called the *symplectic normal space* at m; it is endowed with a natural symplectic structure ω_V inherited from $\omega(m)$. By the compactness of G_μ there is an Ad_{G_μ}-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, relative to which there is the orthogonal direct sum decompositions $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ for some subspace $\mathfrak{q} \subset \mathfrak{g}$. The inner product also allows us to identify all these Lie algebras with their duals. In particular, we have the dual orthogonal direct sum $\mathfrak{g}^* = \mathfrak{g}_\mu^* \oplus \mathfrak{q}^*$, which allows an identification of $\mathfrak{g}_\mu^*$ with a subset of $\mathfrak{g}^*$. The inclusions induced by these identifications are used in Theorem 4.3 whose proof can be found in [7,41] or in the original papers.

Theorem 4.3 (Marle–Guillemin–Sternberg normal form). *Let (M, ω) be as in Theorem 4.1 and $m \in M$, such that $\mu = \mathbf{J}(m) \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$. Then*

$$Y = G \times \mathfrak{g}_\mu{}^* \times V$$

is a Hamiltonian G-space where G acts on Y by $g \cdot (h, \eta, v) = (gh, \eta, v)$ with corresponding equivariant momentum map $\mathbf{J}_Y : Y \to \mathfrak{g}^*$ given by

$$\mathbf{J}_Y(g, \eta, v) = g \cdot (\mu + \eta).$$

In addition, there are G-invariant neighborhoods U of $m \in M$, U' of $(e, 0, 0) \in Y$ and an equivariant symplectic diffeomorphism $\phi : U \to U'$ satisfying $\phi(m) = (e, 0, 0)$ and $\mathbf{J}_Y \circ \phi = \mathbf{J}$.

One of the uses of the MGS normal form is the convenient local characterization of the reduced spaces that is facilitated by the following proposition of Bates and Lerman (see [7] or [41] for a proof).

Proposition 4.4. *In the hypotheses of Theorems 4.1 and 4.3, for a small enough neighborhood Y_0 of the orbit $G \cdot (e, 0, 0)$ in the model space Y, the intersection of the set $\mathbf{J}_Y^{-1}(\mu)$ with the neighborhood Y_0 has the form*

$$\mathbf{J}_Y^{-1}(\mu) \cap Y_0 = \{(g, \eta, v) \in Y_0 \mid g \in G_\mu,\ \eta = 0\}.$$

As promised we have the following.

Theorem 4.5. *The reduced space* (M_μ, ω_μ) *is locally symplectomorphic to* (V, ω_V), *where* V *is the symplectic normal space associated to the* G*-action at* $m \in M$.

Proof (See [48,7,41]). The proof uses Proposition 4.4 in order to show that $\mathbf{J}^{-1}(\mu)$ can be locally represented as $G_\mu \times V$. This implies that $\mathbf{J}^{-1}(\mu)/G_\mu \simeq V$. The construction of the MGS normal form guarantees that this local diffeomorphism is a symplectomorphism if V is endowed with the symplectic form ω_V. □

Remark 4.6. *Note that Proposition* 4.4 *has interesting implications when one looks at the appearance of the dynamical evolution induced by a* G*-invariant Hamiltonian on this model. More specifically, if* F_t *is the Hamiltonian flow associated to the Hamiltonian* h, *then for a time* t *close enough to* 0,

$$F_t(m) \simeq F_t(e, 0, 0) = (g(t), 0, v(t)),$$

for some curve $g(t) \in G_\mu$, *and* $v(t) \in V$. *The zero in the second entry is a direct consequence of Proposition* 4.4. *The fact that* $g(t) \in G_\mu$, *follows from Noether's theorem. If the Hamiltonian* h *is* G*-invariant, the curves* $g(t)$ *and* $v(t)$ *are completely determined by the so-called* reconstruction equations (*see* [41,42], *for a detailed exposition*).

5. Stability of relative periodic orbits

After the background introduced in the previous section we now define the *relative critical elements* of a Hamiltonian system with symmetry $(M, \omega, G, \mathbf{J} : M \to \mathfrak{g}^*, h : M \to \mathbb{R})$. As before, we assume that the symmetry is given by the Lie group G acting freely and properly on M. The Hamiltonian $h \in C^\infty(M)$ is G-invariant and $\mathbf{J}$ is assumed to be equivariant.

Definition 5.1. In the Hamiltonian system with symmetry just described, $m \in M$ is called a *relative periodic point (RPP)*, if there is a $\tau > 0$ and an element $g \in G$ such that

$$F_{t+\tau}(m) = g \cdot F_t(m) \quad \text{for any } t \in \mathbb{R},$$

where F_t is the flow of the Hamiltonian vector field X_h. The set

$$\gamma(m) := \{F_t(m) | t > 0\}$$

is called a *relative periodic orbit (RPO)* through m. The constant $\tau > 0$ is its *relative period* and the group element $g \in G$ is its *phase shift*.

Note the similarity of this definition with the concept of *relative equilibrium*, that is, a point $z \in M$ such that in the reduced space it becomes an equilibrium. An RPO is an orbit of X_h such that in the reduced space it is a periodic orbit. These remarks are made more precise in the following theorem.

Theorem 5.2. *In the conditions of Definition* 5.1, *the following statements are equivalent:*
(i) *the point* m *is an RPP such that* $\mathbf{J}(m) = \mu$;

(ii) *there is a constant* $\tau > 0$ *and* $g \in G_\mu$ *such that*

$$F_{t+\tau}(m) = g \cdot F_t(m) \quad \text{for any } t \in \mathbb{R},$$

where F_t *is the flow of* X_h*;*

(iii) *the point* $[m]_\mu := \pi_\mu(m)$ *is a periodic point of* $(M_\mu, \omega_\mu, h_\mu)$.

Proof. (i) ⇒ (ii). If m is an RPP, there is a $\tau > 0$ such that $F_\tau(m) = g \cdot m$. Applying $\mathbf{J}$ to both sides of this equality and recalling Noether's theorem and the equivariance of $\mathbf{J}$, one obtains that $\mu = g \cdot \mu$, that is $g \in G_\mu$.

(ii) ⇒ (iii). If, with the notation of Theorem 4.1, we apply π_μ on both sides of the equality $F_{t+\tau}(m) = g \cdot F_t(m)$, we obtain that

$$\pi_\mu(F_{t+\tau}(m)) = \pi_\mu(g \cdot F_t(m)) = \pi_\mu(F_t(m)),$$

or equivalently,

$$F^\mu_{t+\tau}([m]_\mu) = F^\mu_t([m]_\mu),$$

where F^μ_t is the flow of the Hamiltonian vector field on M_μ defined by the reduced Hamiltonian function h_μ. This shows that $[m]_\mu$ is a periodic point of $(M_\mu, \omega_\mu, h_\mu)$.

(iii) ⇒ (i). By hypothesis, there is a $\tau > 0$ such that $F^\mu_{t+\tau}([m]_\mu) = F^\mu_t([m]_\mu)$ for any t. Thus

$$\pi_\mu(F_{t+\tau}(m)) = \pi_\mu(F_t(m)) \quad \text{for any } t.$$

In particular, for $t = 0$, $(\pi_\mu \circ F_\tau)(m) = \pi_\mu(m)$ and hence there exists an element $g \in G_\mu$ such that $F_\tau(m) = g \cdot m$. Thus, if t is arbitrary,

$$F_{t+\tau}(m) = (F_t \circ F_\tau)(m) = F_t(g \cdot m) = g \cdot F_t(m),$$

as required. Notice that in the last step we used the G-equivariance of F_t, implied by the G-invariance of h. □

In a Hamiltonian system like the one we are dealing with, the existence of a symmetry gives rise to drift phenomena, making nontrivial the choice of a definition of stability. As it was already the case with relative equilibria (see [47,25,43,36,24,42]), the obvious option, orbital stability, becomes too restrictive. The most natural thing to do is to imitate the notion of stability relative to a subgroup introduced by Patrick [43].

Definition 5.3. In the conditions of Definition 5.1, if G' is a Lie subgroup of G, the RPP m is G'-*stable*, or *stable modulo* G', if for any G'-invariant open neighborhood V of the set $G' \cdot \{F_t(m)\}_{t>0}$, there is an open neighborhood $U \subseteq V$ of m such that $F_t(U) \subset V$, for any $t > 0$.

We can state now the main result on the stability of RPPs.

Theorem 5.4 (The symmetric energy-integrals method). *Let* $(M, \omega, G, \mathbf{J}: M \to \mathfrak{g}^*, h: M \to \mathbb{R})$ *be a Hamiltonian system with a symmetry given by the Lie group* G

acting freely and properly on M. Assume that the Hamiltonian $h \in C^\infty(M)$ is G-invariant and that $\mathbf{J}$ is equivariant. Let $m \in M$ be an RPP such that $\mathbf{J}(m) = \mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$ and G_μ is compact. Then, if there is a set of G_μ-invariant conserved quantities $C^1, \ldots, C^n : M \to \mathbb{R}$ for which

$$\mathbf{d}(C^1 + \cdots + C^n)(m) = 0,$$

and

$$\mathbf{d}^2(C^1 + \cdots + C^n)(m)|_{W\times W}$$

is definite for some (and hence for any) W such that

$$\ker \mathbf{d}C^1(m) \cap \cdots \cap \ker \mathbf{d}C^n(m) \cap T_m\mathbf{J}^{-1}(\mu) = W \oplus (\mathrm{span}\{X_h(m)\} + T_m(G_\mu \cdot m)), \tag{5.1}$$

then m is a G_μ-stable RPP. If $\dim W = 0$ (in particular, if $\dim M_\mu = 2$), then m is always a G_μ-stable RPP.

Proof. We first study the case $\dim W > 0$. Carrying out a computation similar to the one in Theorem 3.1, it is easy to see that the result does not depend on the choice of the point m in the RPO. Moreover, the choice of W is also irrelevant since $\mathbf{d}^2(C^1 + \cdots + C^n)(m)(v, w) = 0$, whenever $v \in \mathrm{span}\{X_h(m)\} + T_m(G_\mu \cdot m)$. Indeed, if we take, without loss of generality, $v = X_h(m) + \xi_M(m) = X_h(m) + X_{\mathbf{J}^\xi}(m)$, with $\xi \in \mathfrak{g}_\mu$, the definition of the Hessian implies that

$$\begin{aligned}\mathbf{d}^2(C^1 + \cdots + C^n)(m)(v, w) &= w[(X_h + X_{\mathbf{J}^\xi})(C^1 + \cdots + C^n)]\\ &= w[\{C^1, h\} + \cdots + \{C^n, h\} + \{C^1, \mathbf{J}^\xi\} + \cdots + \{C^n, \mathbf{J}^\xi\}] = 0,\end{aligned}$$

given that $\{C^i, h\} = 0$ since C^i is a conserved quantity for $i \in \{1, \ldots, n\}$, and $\{C^i, \mathbf{J}^\xi\}(z) = \mathbf{d}C^i(z) \cdot \xi_M(z) = 0$, for any $z \in M$, since C^i is G_μ-invariant. The G_μ-invariance of the conserved quantities $C^1, \ldots, C^n$, when restricted to $\mathbf{J}^{-1}(\mu)$, implies the existence of the functions ${C^1}_\mu, \ldots, {C^n}_\mu : M_\mu \to \mathbb{R}$, uniquely defined by the relation ${C^i}_\mu \circ \pi_\mu = C^i \circ i_\mu$, with $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$ the natural inclusion of Theorem 4.1, and $i \in \{1, \ldots, n\}$. Because $\mathbf{d}(C^1 + \cdots + C^n)(m) = 0$, we necessarily have that

$$\begin{aligned}0 &= ({i_\mu}^*\mathbf{d}(C^1 + \cdots + C^n))(m) = \mathbf{d}((C^1 + \cdots + C^n) \circ i_\mu)(m)\\ &= \mathbf{d}(C^1 \circ i_\mu + \cdots + C^n \circ i_\mu)(m) = \mathbf{d}({C^1}_\mu \circ \pi_\mu + \cdots + {C^n}_\mu \circ \pi_\mu)(m)\\ &= \mathbf{d}({C^1}_\mu + \cdots + {C^n}_\mu)([m]_\mu) \circ T_m\pi_\mu.\end{aligned}$$

Since π_μ is a surjective submersion, this implies that

$$\mathbf{d}({C^1}_\mu + \cdots + {C^n}_\mu)([m]_\mu) = 0. \tag{5.2}$$

Recall that by part (iv) of Theorem 4.1

$$X_{h_\mu}([m]_\mu) = T_m\pi_\mu \cdot X_h(m).$$

Hence, applying $T_m\pi_\mu$ on both sides of (5.1) one obtains

$$\ker \mathbf{d}{C^1}_\mu([m]_\mu) \cap \cdots \cap \ker \mathbf{d}{C^n}_\mu([m]_\mu) = T_m\pi_\mu(W) \oplus \operatorname{span}\{X_{h_\mu}([m]_\mu)\}. \quad (5.3)$$

Note that the sum in (5.3) is direct because if there was a $w \in W$ such that $T_m\pi_\mu(w) = T_m\pi_\mu(X_h(m))$, then $w - X_h(m)$ would be in the kernel of $T_m\pi_\mu$ and therefore there would exist an element $\xi \in \mathfrak{g}_\mu$ such that $w - X_h(m) = \xi_M(m)$. This would imply that $w \in W \cap (\operatorname{span}\{X_h(m)\} + T_m(G_\mu \cdot m))$ which, by the definition of W, implies that $w = 0$.

In the language of Theorem 3.1, expression (5.3) shows that $T_m\pi_\mu(W)$ is a complement to $\operatorname{span}\{X_{h_\mu}([m]_\mu)\}$ in $\ker \mathbf{d}{C^1}_\mu([m]_\mu) \cap \cdots \cap \ker \mathbf{d}{C^n}_\mu([m]_\mu)$ for the periodic point $[m]_\mu$. Moreover, (5.2) implies that $\mathbf{d}^2({C^1}_\mu + \cdots + {C^n}_\mu)([m]_\mu)$ is well-defined. Using Proposition 2.5 we write

$$\begin{aligned}
&\mathbf{d}^2({C^1}_\mu + \cdots + {C^n}_\mu)([m]_\mu)(T_m\pi_\mu \cdot w_1, T_m\pi_\mu \cdot w_2)\\
&= \mathbf{d}^2({\pi_\mu}^*({C^1}_\mu + \cdots + {C^n}_\mu))(m)(w_1, w_2)\\
&= \mathbf{d}^2(({C^1}_\mu + \cdots + {C^n}_\mu) \circ \pi_\mu)(m)(w_1, w_2)\\
&= \mathbf{d}^2((C^1 + \cdots + C^n) \circ i_\mu)(m)(w_1, w_2)\\
&= \mathbf{d}^2({i_\mu}^*(C^1 + \cdots + C^n))(m)(w_1, w_2)\\
&= \mathbf{d}^2(C^1 + \cdots + C^n)(m)(w_1, w_2).
\end{aligned}$$

Since $w_1, w_2 \in W$ are arbitrary, this equality and the hypothesis of the theorem guarantees that

$$\mathbf{d}^2({C^1}_\mu + \cdots + {C^n}_\mu)([m]_\mu)|_{T_m\pi_\mu \cdot W \times T_m\pi_\mu \cdot W}$$

is a definite quadratic form. Therefore the periodic point $[m]_\mu$ satisfies the hypothesis of Theorem 3.1, which implies the orbital stability of γ_μ, the periodic orbit in M_μ through $[m]_\mu$.

Our next step will be to show that this orbit is also stable in the space M/G_μ. Since G_μ is compact and acts freely on M, by Theorem 4.2, M/G_μ is a Poisson manifold and the canonical projection $\pi : M \to M/G_\mu$ is a Poisson map. The point $[m] := \pi(m)$ is clearly periodic with respect to the Poisson dynamics induced by h on M/G_μ. We will denote by γ the periodic orbit associated to $[m] \in M/G_\mu$. Since $\mathbf{J}^{-1}(\mu)$ is a regular submanifold of M, the reduced space M_μ is a regular submanifold of M/G_μ. Of course, $[m] = [m]_\mu$ and $\gamma = \gamma_\mu$ but we want to distinguish in what follows between objects in M/G_μ and M_μ.

We now construct, with the help of the normal form introduced in Theorem 4.3, a local transversal section S in M/G_μ to the closed orbit γ at $[m]$, such that $S_\mu = S \cap M_\mu$ is a local submanifold of M_μ and moreover, it is a local transversal section to γ_μ at $[m]_\mu$. The MGS normal form on M states that a G-invariant open neighborhood of $m \in M$ is G-equivariantly diffeomorphic to a G-invariant neighborhood of $(e, 0, 0) \in Y = G \times {\mathfrak{g}_\mu}^* \times V$ and that this diffeomorphism maps $m \in M$ to $(e, 0, 0) \in Y$. This implies that in a neighborhood of $[m]$, M/G_μ is locally diffeomorphic to an open neighborhood of $([e], 0, 0)$ in $G/G_\mu \times {\mathfrak{g}_\mu}^* \times V$. In addition, by Theorem 4.5, the space M_μ can be locally identified around $[m]_\mu$ with $\{[e]\} \times \{0\} \times V \simeq V$. Let now $S_\mu \subset V \simeq M_\mu$ be a local transversal section to γ_μ at $[m]_\mu$ and let S be the local codimension one submanifold in M/G_μ around $[m]$ given by

$$S := G/G_\mu \times \mathfrak{g}_\mu{}^* \times S_\mu.$$

By construction, S is a local transversal section to γ at $[m]$ such that

$$S_\mu = S \cap M_\mu.$$

If we now use S_μ, together with $\mathbf{d}(C^1{}_\mu + \cdots + C^n{}_\mu)([m]_\mu) = 0$ and the definiteness of

$$\mathbf{d}^2(C^1{}_\mu + \cdots + C^n{}_\mu)([m]_\mu)|_{T_m\pi_\mu(W)\times T_m\pi_\mu(W)}$$

we can repeat the first part of the proof of Theorem 3.1 in order to prove the existence of a constant $a > 0$ (see (3.1)) for which the map $f_\mu : M_\mu \to \mathbb{R}$ given by

$$\begin{aligned} f_\mu := &a(C^1{}_\mu - C^1{}_\mu([m]_\mu) + \cdots + C^n{}_\mu - C^n{}_\mu([m]_\mu)) \\ &+(C^1{}_\mu - C^1{}_\mu([m]_\mu))^2 + \cdots + (C^n{}_\mu - C^n{}_\mu([m]_\mu))^2 \end{aligned}$$

satisfies $\mathbf{d}f_\mu([m]_\mu) = 0$ and $\mathbf{d}^2(f_\mu|_{S_\mu})([m]_\mu)$ is positive definite.

Note also that the G_μ-invariance of h and $C^1, \ldots, C^n$ implies the existence of functions $[h], [C^1], \ldots, [C^n] \in C^\infty(M/G_\mu)$, uniquely determined by the relations

$$[h] \circ \pi = h \quad \text{and} \quad [C^i] \circ \pi = C^i.$$

We clearly have $[h]|_{M_\mu} = h_\mu, [C^i]|_{M_\mu} = C^i_\mu$ for $i \in \{1, \ldots n\}$. Let f be the extension of f_μ to M/G_μ, defined by

$$\begin{aligned} f = &a([C^1] - [C^1]([m]) + \cdots + [C^n] - [C^n]([m])) \\ &+([C^1] - [C^1]([m]))^2 + \cdots + ([C^n] - [C^n]([m]))^2. \end{aligned}$$

To make this extension f of f_μ to M/G_μ completely explicit, we shall use the notation introduced in the following commutative diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) & \xrightarrow{i_\mu} & M \\ {\scriptstyle \pi_\mu}\downarrow & & \downarrow{\scriptstyle \pi} \\ M_\mu & \xrightarrow{i_\mu^{M_\mu}} & M/G_\mu. \end{array}$$

and observe that

$$f \circ i_\mu{}^{M_\mu} = f_\mu.$$

Moreover, since h and $C^1, \ldots, C^n$ are conserved quantities for X_h, f is a conserved quantity for $X_{[h]}$, the vector field induced by $[h]$ on M/G_μ defined via its Poisson structure. Note that by construction,

$$\begin{aligned} f \circ \pi = &a(C^1 - C^1(m) + \cdots + C^n - C^n(m)) \\ &+(C^1 - C^1(m))^2 + \cdots + (C^n - C^n(m))^2. \end{aligned}$$

Since $\mathbf{d}\left((C^1 - C^1(m))^2 + \cdots + (C^n - C^n(m))^2\right)(m) = 0$ and $C^1 + \cdots + C^n$ differs from $C^1 - C^1(m) + \cdots + C^n - C^n(m)$ by a constant, it is clear that $\mathbf{d}(f \circ \pi)(m) = 0$, which implies that for any $v \in T_m M$

$$\mathbf{d}f([m])(T_m\pi \cdot v) = \mathbf{d}(\pi^* f)(m) \cdot v = \mathbf{d}(f \circ \pi)(m) \cdot v = 0.$$

As π is a surjective submersion, this implies that $\mathbf{d}f([m]) = 0$ and hence $\mathbf{d}^2 f([m])$ is well defined.

Let now Z be the vector subspace of $T_{[m]}(M/G_\mu)$ defined by

$$Z := T_{[m]}S \cap T_{[m]_\mu} i_\mu{}^{M_\mu}(T_{[m]}M_\mu),$$

or, locally in terms of the MGS normal form,

$$Z := \{0\} \times \{0\} \times T_{[m]_\mu} S_\mu.$$

We now show that $\mathbf{d}^2(f|_S)([m])|_{Z\times Z}$ is positive definite. We will use the notation introduced in the following commutative diagram:

$$\begin{array}{ccc} S_\mu & \xrightarrow{i_{S_\mu}} & M_\mu \\ {\scriptstyle i_\mu^{S_\mu}}\downarrow & & \downarrow{\scriptstyle i_\mu^{M_\mu}} \\ S & \xrightarrow{i_S} & M/G_\mu. \end{array}$$

Let $v_1, v_2 \in Z$. These vectors can be written as $v_i = T_{[m]_\mu} i_\mu{}^{S_\mu} \cdot w_i$, with $w_i \in T_{[m]_\mu} S_\mu$, $i \in \{1, 2\}$. Since

$$\begin{aligned} \mathbf{d}^2(f|_S)([m])(v_1, v_2) &= \mathbf{d}^2(f|_S)(i_\mu{}^{S_\mu}([m]_\mu))(T_{[m]_\mu} i_\mu{}^{S_\mu} \cdot w_1, T_{[m]_\mu} i_\mu{}^{S_\mu} \cdot w_2) \\ &= \mathbf{d}^2((i_S \circ i_\mu{}^{S_\mu})^* f)([m]_\mu)(w_1, w_2) \\ &= \mathbf{d}^2((i_\mu{}^{M_\mu} \circ i_{S_\mu})^* f)([m]_\mu)(w_1, w_2) \\ &= \mathbf{d}^2(f_\mu|_{S_\mu})([m]_\mu)(w_1, w_2), \end{aligned}$$

the positive definiteness of $\mathbf{d}^2(f_\mu|_{S_\mu})([m]_\mu)$ ensures that $\mathbf{d}^2(f|_S)([m])|_{Z\times Z}$ is positive definite.

Keeping this in mind, we now define a real-valued function on M/G_μ with the help of an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}^*$, invariant under the coadjoint action of G_μ (recall that G_μ is compact by hypothesis). If we denote by $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ the associated norm, we define the map:

$$\begin{aligned} j : M/G_\mu &\longrightarrow \mathbb{R} \\ [z] &\longmapsto (|\mathbf{J}(z)| - |\mu|)^2. \end{aligned}$$

Locally, around $[m] \simeq ([e], 0, 0)$, j can be expressed in terms of the MGS normal form as $j([g], \eta, v) = (|\mathrm{Ad}^*{}_{g^{-1}}(\mu + \eta)| - |\mu|)^2$. This map is well-defined because if $z' = g \cdot z$ for some $g \in G_\mu$, the equivariance of $\mathbf{J}$ and $\mathrm{Ad}^*_{G_\mu}$-invariance of $\langle\cdot,\cdot\rangle$, guarantee that

$$(|\mathbf{J}(z')| - |\mu|)^2 = (|\mathbf{J}(g \cdot z)| - |\mu|)^2 = (|g \cdot \mathbf{J}(z)| - |\mu|)^2 = (|\mathbf{J}(z)| - |\mu|)^2.$$

It is easy to see that j has a critical point at $[m]$, that is $\mathbf{d}j[m] = 0$. Moreover, we will now show that $\mathbf{d}^2(j|_S)([m])$ is positive semidefinite with kernel Z. Indeed, if $[v_i] = T_m\pi \cdot v_i \in T_{[m]}S$ with $v_i \in T_mM$ and $i \in \{1, 2\}$, suppose that

$$\mathbf{d}^2(j|_S)([m])([v_1], [v_2]) = 0 \quad \text{for all } [v_2] \in T_{[m]}S. \tag{5.4}$$

The definition of j immediately implies that

$$\mathbf{d}^2(j|_S)([m])([v_1], [v_2]) = 2|T_m\mathbf{J} \cdot v_1|\,|T_m\mathbf{J} \cdot v_2|.$$

Since equality (5.4) holds for all $[v_2] \in T_{[m]}S$, it holds in particular for $[v_2] = [v_1]$. In that case we have $|T_m\mathbf{J} \cdot v_1| = 0$ which implies that $T_m\mathbf{J} \cdot v_1 = 0$, that is, $v_1 \in \ker T_m\mathbf{J}$. Let us write this conclusion in terms of the MGS normal form. In general, there are elements $\sigma \in \mathfrak{g}$, $\eta \in \mathfrak{g}_\mu{}^*$, and $v \in S_\mu$ such that

$$v_1 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp t\sigma, t\eta, tv).$$

Then,

$$0 = T_m\mathbf{J} \cdot v_1 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*{}_{\exp(-t\sigma)}(\mu + t\eta) = -\mathrm{Ad}^*{}_\sigma\mu + \eta.$$

However, note that $\mathrm{Ad}_\sigma{}^*\mu \in \mathfrak{g}_\mu{}^\circ$, the annihilator of $\mathfrak{g}_\mu$ in $\mathfrak{g}^*$, and $\eta \in \mathfrak{g}_\mu^*$. Therefore, $\mathrm{Ad}_\sigma{}^*\mu = 0$ and $\eta = 0$ since $\mathfrak{g}_\mu{}^\circ \cap \mathfrak{g}_\mu^* = \{0\}$ in $\mathfrak{g}^*$. Hence,

$$v_1 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp t\sigma, 0, tv),$$

with $\sigma \in \mathfrak{g}_\mu$ and $v \in S_\mu$, which implies that $[v_1] = ([0], 0, v) \in Z$, as required.

Putting together what we know about f and j, Lemma 2.6 guarantees the existence of a constant $b > 0$ such that the map, $F := bf + j$ is such that $\mathbf{d}^2(F|_S)[m]$ is positive definite. Since both f and j are constants of the motion for the Poisson vector field $X_{[h]}$, so is F. Using this map, it can be shown by repeating the last stages of the proof of Theorem 3.1, that $[m]$ is a stable periodic point and hence γ is orbitally stable. (Recall that the topology of M/G_μ is metrizable, since it is induced by the smooth structure of M/G_μ as a regular quotient manifold (see [11], Proposition 9.4.1), a technical device used in the proof of Theorem 3.1).

Once the orbital stability of γ has been proven, it is straightforward to show the G_μ-stability of m as an RPP. If V is a G_μ-invariant open neighborhood of $G_\mu \cdot \{F_t(m)\}_{t>0}$, then $\pi(V)$ is an open neighborhood of γ in M/G_μ (since the canonical projection π is a submersion, it is an open map ([11], Proposition 6.1.5)). The orbital stability of γ in M/G_μ implies the existence of another open neighborhood U of $[m] \in M/G_\mu$ such that $U \subseteq \pi(V)$ and $F_t^{[h]}(U) \subseteq \pi(V)$ for all $t > 0$, where $F_t^{[h]}$ denotes the flow of $X_{[h]}$, characterized by the equality:

$$\pi \circ F_t = F_t^{[h]} \circ \pi.$$

We will use this identity in order to prove that $\pi^{-1}(U)$ is the open set that we are looking for in order to conclude the G_μ-stability of m. Since π is surjective and $U \subseteq \pi(V)$ we have that

$$\pi^{-1}(U) \subseteq \pi^{-1}(\pi(V)) = V.$$

We now show that $F_t(\pi^{-1}(U)) \subseteq V$ for positive time. If $u \in M$ is such that $\pi(u) \in U$, we know that

$$F_t^{[h]}(\pi(u)) \in \pi(V) \text{ for all } t \quad \Longleftrightarrow \quad \pi(F_t(u)) \in \pi(V) \text{ for all } t.$$

Hence, for any $t > 0$ there is a $g(t) \in G_\mu$ such that $F_t(u) = g(t) \cdot v$; but since V is G_μ-invariant, $g(t) \cdot v = F_t(u) \in V$, as required and the proof of G_μ-stability of m as an RPP is finished.

We now consider the case $W = \{0\}$. The proof in this case is identical if we take $f_\mu = (C^1{}_\mu - C^1{}_\mu([m]_\mu))^2 + \cdots + (C^n{}_\mu - C^n{}_\mu([m]_\mu))^2$ which, in this particular case satisfies $\mathbf{d}f_\mu([m]_\mu) = 0$ and $\mathbf{d}^2(f_\mu|_{S_\mu})([m]_\mu)$ is positive definite. Note that $W = \{0\}$ includes the case $\dim M_\mu = 2$ since, by the relation (5.3):

$$\ker \mathbf{d}C^1{}_\mu([m]_\mu) \cap \cdots \cap \ker \mathbf{d}C^n{}_\mu([m]_\mu) = T_m\pi_\mu(W) \oplus \operatorname{span}\{X_{h_\mu}([m]_\mu)\}.$$

Thus, if $\dim M_\mu = 2$, then necessarily $\dim T_m\pi_\mu(W) = 0$ which implies that $W \subset T_m(G_\mu \cdot m)$, and so, by construction, $W = \{0\}$. □

Example 5.5. We consider several simple elementary examples to illustrate the use of Theorem 5.4.

(i) *The S^1-stability of the RPOs of the spherical pendulum:* The spherical pendulum consists of a particle of mass m, moving under the action of a constant gravitational field of acceleration g, on the surface of a sphere of radius l. If we use spherical coordinates with origin the center of the sphere and polar axis pointing vertically downwards, the Lagrangian of this system is

$$L(\theta, \varphi, \dot\theta, \dot\varphi) = \tfrac{1}{2}ml^2(\dot\theta^2 + \dot\varphi^2 \sin^2\theta) + mgl\cos\theta. \tag{5.5}$$

The solution of the Euler–Lagrange equations that follow from (5.5) is a classical problem on elliptic functions whose solution shows that, generically, the motion of the bob describes RPOs in the phase space with respect to the S^1 symmetry of the problem. We will show that these RPOs are stable modulo S^1.

In order to use Theorem 5.4, we use the Legendre transform to write the system down in phase space variables $(\theta, \varphi, p_\theta, p_\varphi)$, where the canonical symplectic form is $\Omega = \mathbf{d}\theta \wedge \mathbf{d}p_\theta + \mathbf{d}\varphi \wedge \mathbf{d}p_\varphi$ and the Hamiltonian of the spherical pendulum can be written as

$$h(\theta, \varphi, p_\theta, p_\varphi) = \frac{p_\theta^2}{2ml^2} + \frac{p_\varphi^2}{2ml^2\sin^2\theta} - mgl\cos\theta.$$

It may be readily verified that this system is invariant under the lifted action to T^*S^2 of $SO(2)$ on S^2 by φ-rotations. This action has the well-known associated equivariant momentum map

$$\mathbf{J} : T^*S^2 \longrightarrow \mathfrak{so}(2)^* \simeq \mathbb{R}$$
$$(\theta, \varphi, p_\theta, p_\varphi) \longmapsto p_\varphi.$$

We will restrict ourselves to regular values of $\mathbf{J}$, that is, we will choose certain $\mu \neq 0$ in $\mathfrak{so}(2)^*$, and we will reduce at it. Clearly,

$$\mathbf{J}^{-1}(\mu) = \{(\theta, \varphi, p_\theta, \mu) | (\theta, \varphi, p_\theta, \mu) \in T^*S^2\}$$

and, since $SO(2)$ is abelian, $G_\mu = SO(2)$ acts via

$$SO(2) \times \mathbf{J}^{-1}(\mu) \longrightarrow \mathbf{J}^{-1}(\mu),$$
$$(\mathrm{e}^{\mathrm{i}\alpha}, (\theta, \varphi, p_\theta, \mu)) \longmapsto (\theta, \varphi + \alpha, p_\theta, \mu).$$

The reduced space $M_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ can be naturally identified with $T^*S^1_+$, where S^1_+ is the upper semicircle, by taking as canonical projection

$$\pi_\mu : \mathbf{J}^{-1}(\mu) \longrightarrow M_\mu$$
$$(\theta, \varphi, p_\theta, \mu) \longmapsto (\theta, p_\theta).$$

The reduced symplectic form ω_μ, uniquely determined by the relation $i_\mu{}^*\omega = \pi_\mu{}^*\omega_\mu$, takes the form, $\omega_\mu = \mathbf{d}\theta \wedge \mathbf{d}p_\theta$. The Hamiltonian reduces to

$$h_\mu(\theta, p_\theta) = \frac{p_\theta^2}{2ml^2} + \frac{\mu^2}{2ml^2 \sin^2\theta} - mgl\cos\theta,$$

which implies that $(M_\mu, \omega_\mu, h_\mu)$ is a simple mechanical system (its Hamiltonian has the form kinetic+potential) with potential energy given by

$$V_\mu(\theta) = \frac{\mu^2}{2ml^2 \sin^2\theta} - mgl\cos\theta.$$

In the classical literature (see [16,22]), $V_\mu(\theta)$ is called the *effective potential* of the reduced problem. Fig. 4 exhibits its main features, which allow us to classify the different kinds of motions that the system may generate in terms of the value of its total energy.

Note that $V_\mu(\theta)$ has a single minimum θ_0, between 0 and π, determined by the relation

$$mgl\sin\theta_0 - \frac{\mu^2}{ml^2}\cot\theta_0 \csc^2\theta_0 = 0.$$

If the total energy of the system equals $V_\mu(\theta_0) := E_{\mathrm{circ}}$, the pendulum describes a circular orbit of radius $l\sin\theta_0$, whose stability can be studied using the energy-momentum method (see [47,25,43,36,24,42]). If the total energy of the system E is such that $E > E_{\mathrm{circ}}$, the motion of the pendulum is bounded in its θ coordinate between certain limit values $\theta_{\min}(E)$ and $\theta_{\max}(E)$, uniquely determined by the relation $V_\mu(\theta_{\min}(E)) = V_\mu(\theta_{\max}(E)) = E$; moreover, the motion in the reduced space M_μ is periodic as we will prove below.

Proposition 5.6. *Let (M, ω, h) be a two-dimensional Hamiltonian system and let $m \in M$ be a point such that $h(m) = E$, with E a regular value of the Hamiltonian h, such that the*

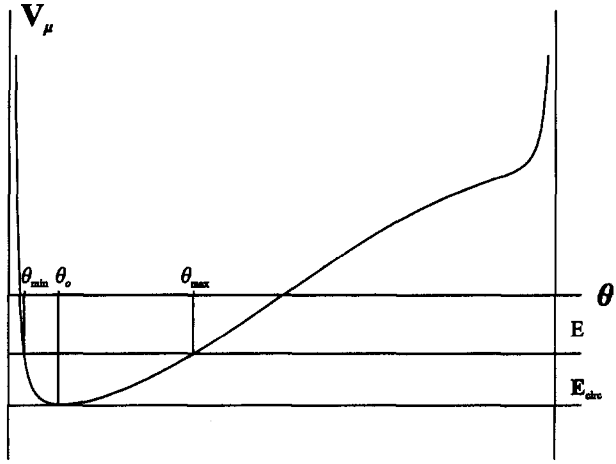

Fig. 4. Typical effective potential of the spherical pendulum.

connected component of $h^{-1}(E)$ that contains m is compact. Then, m is a periodic point, that is, there is a $\tau > 0$ for which $F_\tau(m) = m$, where F_t is the Hamiltonian flow generated by h.

Proof. Since every one-dimensional compact and connected manifold is diffeomorphic to a circle, so is the case for the connected component of $h^{-1}(E)$ that contains m. The regularity of E implies that this circle does not contain equilibria and therefore, by the uniqueness of the flow, the time evolution on it must be periodic. □

We now apply this result on the reduced space $(M_\mu, \omega_\mu, h_\mu)$. If we choose $[m]_\mu = (\theta, p_\theta)$, such that $h_\mu([m]_\mu) = E > E_{\mathrm{circ}}$, the hypothesis of Proposition 5.6 on the regularity of E is satisfied. We now show that ${h_\mu}^{-1}(E)$ is compact in M_μ. Clearly ${h_\mu}^{-1}(E)$ is closed. So all we need to show is its boundedness. As we already know, the θ variable is bounded between certain limit values $\theta_{\min}(E)$ and $\theta_{\max}(E)$. By conservation of energy, θ and p_θ are related by

$$E = \frac{p_\theta^2}{2ml^2} + \frac{\mu^2}{2ml^2 \sin^2\theta} - mgl\cos\theta$$

or, equivalently

$$p_\theta = \pm\sqrt{2ml^2\left(E - \frac{\mu^2}{2ml^2\sin^2\theta} + mgl\cos\theta\right)}. \tag{5.6}$$

Since p_θ is a continuous function of θ defined on the compact set $[\theta_{\min}, \theta_{\max}]$, strictly included on $[0, \pi]$, it reaches a minimum and a maximum and, therefore it is bounded in $h^{-1}(E)$. Since $h^{-1}(E)$ is closed and bounded, it is compact and, by Proposition 5.6, the Hamiltonian flow corresponding to h_μ on this range of energies consists of periodic orbits. These periodic orbits lift, by Theorem 5.2, to RPOs in T^*S^2. Since $\dim M_\mu = 2$, Theorem 5.4 guarantees that these orbits are $SO(2)$-stable.

(ii) *Stability of the nutating motion of the Lagrange top:* The Lagrange top is an axisymmetric rigid body with a fixed point, moving steadily in a constant gravitational field of acceleration g. We will denote by (I_1, I_1, I_3) its principal moments of inertia, by m its mass, and by l the distance between its center of mass and the fixed point. The phase space for the Lagrange top as a Hamiltonian system is $T^*SO(3)$. If we use the Euler angles (θ, φ, ψ) to parameterize $SO(3)$, and denote by $(p_\theta, p_\varphi, p_\psi)$ the conjugate momenta, the corresponding Hamiltonian function in this chart on $T^*SO(3)$ has the expression

$$h(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi) = \frac{p_\theta^2}{2I_1} + \frac{(p_\varphi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3} + mgl\cos\theta.$$

Denote by ω the canonical symplectic form in $T^*SO(3)$. It can be readily verified that the Hamiltonian system $(T^*SO(3), \omega, h)$ is invariant under the lifted action of the group $S^1 \times S^1$ over $SO(3)$ given by

$$\begin{array}{ccc} (S^1 \times S^1) \times SO(3) & \longrightarrow & SO(3), \\ ((e^{i\phi_1}, e^{i\phi_2}), (\theta, \varphi, \psi)) & \longmapsto & (\theta, \varphi + \phi_1, \psi + \phi_2). \end{array}$$

This action is Hamiltonian and has an associated momentum map given by

$$\begin{array}{rccc} \mathbf{J}: & T^*SO(3) & \longrightarrow & \mathrm{Lie}(S^1 \times S^1) \simeq \mathbb{R}^2 \\ & (\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi) & \longmapsto & (p_\varphi, p_\psi). \end{array}$$

Using this symmetry we will proceed in a fashion similar to the spherical pendulum. Firstly, we will reduce the system at regular values of $\mathbf{J}$, that is, we will restrict ourselves to values $\mu = (\rho, \nu) \in \mathbb{R}^2$ of $\mathbf{J}$ such that $\rho \neq 0$ and $\nu \neq 0$. Clearly,

$$\mathbf{J}^{-1}(\mu) = \{(\theta, \varphi, \psi, p_\theta, \rho, \nu) \mid (\theta, \varphi, \psi, p_\theta, \rho, \nu) \in T^*SO(3)\}$$

and, since $S^1 \times S^1$ is abelian, it follows that $G_\mu = S^1 \times S^1$. Thus, the reduced space $(M_\mu, \omega_\mu, h_\mu)$ can be naturally identified with $T^*S^1_+$, with S^1_+ the open upper semicircle, by taking as the canonical projection

$$\begin{array}{rccc} \pi_\mu: & \mathbf{J}^{-1}(\mu) & \longrightarrow & M_\mu \\ & (\theta, \varphi, \psi, p_\theta, \rho, \nu) & \longmapsto & (\theta, p_\theta). \end{array}$$

With this identification, $\omega_\mu = \mathbf{d}\theta \wedge \mathbf{d}p_\theta$, and

$$h_\mu(\theta, p_\theta) = \frac{p_\theta^2}{2I_1} + \frac{(\rho - \nu\cos\theta)^2}{2I_1 \sin^2\theta} + \frac{\nu^2}{2I_3} + mgl\cos\theta.$$

Analogously to the spherical pendulum, this is a simple mechanical system, with potential energy (effective potential) given by

$$V_\mu(\theta) = \frac{(\rho - \nu\cos\theta)^2}{2I_1 \sin^2\theta} + \frac{\nu^2}{2I_3} + mgl\cos\theta.$$

As can be seen in the Fig. 5, $V_\mu(\theta)$ has a single minimum, θ_0, between 0 and π. When we tune the energy of the system to the value $E_{\mathrm{circ}} := V_\mu(\theta_0)$ the system falls into a

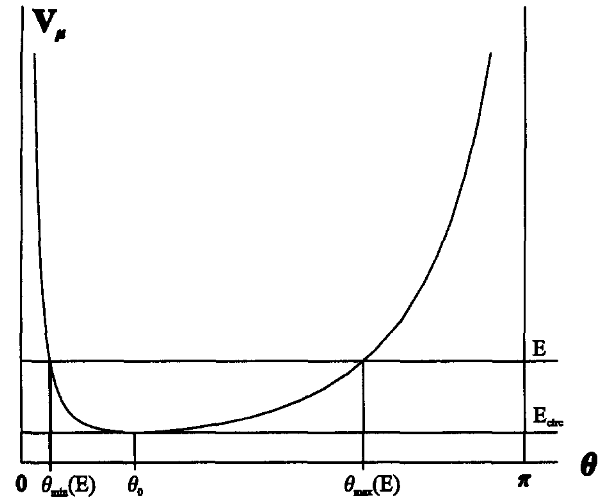

Fig. 5. Typical effective potential for the Lagrange top.

relative equilibrium with respect to the $S^1 \times S^1$ symmetry, whose stability can be studied using the energy-momentum method (see [26,23,42]). If the energy is strictly higher than $E_{\rm circ}$, the variable θ is bounded between certain values $\theta_{\rm min}(E)$ and $\theta_{\rm max}(E)$ for which $V_\mu(\theta_{\rm min}(E)) = V_\mu(\theta_{\rm max}(E)) = E$, and the system describes an RPO, as we prove by showing that the motion in M_μ is periodic using a method identical to the one followed in the case of the spherical pendulum and based on Proposition 5.6. The only difference is that in this case, θ and p_θ are related by

$$p_\theta = \pm\sqrt{2I_1(E - V_\mu(\theta))}.$$

These periodic orbits lift by Theorem 5.2, to RPOs in $T^*SO(3)$. Since $\dim M_\mu = 2$, Theorem 5.4 guarantees that these orbits are $S^1 \times S^1$-stable.

(iii) *Stability of the bounded Manev orbits. Relative periodic motions in central potentials:* It is well-known that due to general relativistic corrections, even in the two body approximation, the planets do not follow Kepler's First Law, that is, their orbits do not describe ellipses but precessing ellipses. In a first approximation, this correction has the form B/r^2, for some constant B, that is, truncating negligible terms, the gravitational potential takes the form

$$V(r) = \frac{k}{r} + \frac{B}{r^2}. \tag{5.7}$$

The introduction of potentials of this form to describe the gravitational motion goes back to Newton and Clairaut (see [13] for excellent historical remarks). However, it was Manev [28–31] who, using physical principles, more specifically, a generalized action–reaction principle, was the first to propose a potential like (5.7) as a correction to the classical Newtonian potential useful in celestial mechanics. The Hamiltonian flow induced by (5.7) has been extensively studied, and completely classified in [12,13,21]; moreover, Diacu et al. [13] have shown that in a certain approximative regime (what they call the solar-system approximation), Manev's model is the natural classical analog of the Schwarzschild problem. In fact, using the values for k and B given by Manev, one obtains an accurate description of the apsidal motion of the moon and the perihelion advance of Mercury. These values are:

$$k = GM, \qquad B = \frac{GM\gamma}{2}, \qquad \gamma = \frac{3GM}{c^2},$$

where G is the constant of gravitation, c the speed of light, and M the mass of the particle at the origin; the mass of the rotating particle is taken to be one.

One of the conclusions of [12] is that the bounded motions of this problem, that is, the solutions with negative energy, are generically precessing ellipses. We will concentrate on this case, and we will show that this part of the flow consists generically of G_μ-stable RPOs.

The phase space for this problem, as a Hamiltonian system, is $T^*\mathbb{R}^3$. If we parameterize $\mathbb{R}^3$ using spherical coordinates (r, θ, φ) (θ denotes the colatitude and φ the azimuth), the corresponding Hamiltonian function of the system can be written as

$$h(r, \theta, \varphi, p_r, p_\theta, p_\varphi) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2 \sin^2\theta} - \frac{k}{r} - \frac{B}{r^2},$$

where m denotes the reduced mass of the two bodies $m = M/(M+1)$. This system is invariant under the lifted action of $SO(3)$ to $T^*\mathbb{R}^3$. Moreover, this action is Hamiltonian with equivariant momentum map given by the angular momentum of the system, that is, $J(\boldsymbol{r}, \boldsymbol{p}) = \boldsymbol{r} \times \boldsymbol{p}$, whose expression in spherical coordinates is

$$\begin{aligned} &\mathbf{J}(r, \theta, \varphi, p_r, p_\theta, p_\varphi) \\ &\quad = (-p_\varphi \cos\varphi \cot\theta - p_\theta \sin\varphi,\ p_\theta \cos\varphi - p_\varphi \cot\theta \sin\varphi,\ p_\varphi). \end{aligned}$$

Given that Theorem 5.4 is valid only for regular values of the momentum map, we will restrict ourselves to values $\mu \neq 0$ of $\mathbf{J}$. More specifically, we will choose our coordinate system in such a fashion that, without loss of generality, μ has the form $\mu = (0, 0, l)$ with $l \neq 0$. It is easy to see that

$$\mathbf{J}^{-1}(\mu) = \left\{ \left(r, \frac{\pi}{2}, \varphi, p_r, 0, l\right) \middle| \left(r, \frac{\pi}{2}, \varphi, p_r, 0, l\right) \in T^*\mathbb{R}^3 \text{ and } \mu = (0, 0, l) \right\}.$$

The coadjoint isotropy subgroup G_μ of μ is isomorphic to $SO(2)$ and acts on $\mathbf{J}^{-1}(\mu)$ as

$$\begin{array}{ccc} G_\mu \times \mathbf{J}^{-1}(\mu) & \longrightarrow & \mathbf{J}^{-1}(\mu) \\ \left(e^{i\alpha}, \left(r, \frac{\pi}{2}, \varphi, p_r, 0, l\right)\right) & \longmapsto & \left(r, \frac{\pi}{2}, \varphi + \alpha, p_r, 0, l\right). \end{array}$$

Hence, $M_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$, can be naturally identified with $T^*\mathbb{R}^+$, by taking as the canonical projection

$$\begin{array}{rccc} \pi_\mu : & \mathbf{J}^{-1}(\mu) & \longrightarrow & M_\mu \\ & \left(r, \frac{\pi}{2}, \varphi, p_r, 0, l\right) & \longmapsto & (r, p_r). \end{array}$$

Moreover, with this identification, the reduced Hamiltonian is

$$h_\mu(r, p_r) = \frac{p_r^2}{2m} + \frac{l^2}{2mr^2} - \frac{k}{r} - \frac{B}{r^2} = \frac{p_r^2}{2m} - \frac{k}{r} + \frac{l^2 - 2mB}{2mr^2}. \tag{5.8}$$

As we know, the reduced symplectic form ω_μ is uniquely determined by the relation

$$\mathrm{i}_\mu{}^*\omega = \pi_\mu{}^*\omega_\mu.$$

Since in spherical coordinates ω is given by $\omega = \mathbf{d}r \wedge \mathbf{d}p_r + \mathbf{d}\theta \wedge \mathbf{d}p_\theta + \mathbf{d}\varphi \wedge \mathbf{d}p_\varphi$ it follows that

$$\omega_\mu = \mathbf{d}r \wedge \mathbf{d}p_r.$$

This implies that $(M_\mu, \omega_\mu, h_\mu)$ is a simple mechanical system with potential energy (effective potential):

$$V_\mu(r) = -\frac{k}{r} + \frac{l^2 - 2mB}{2mr^2},$$

that is, the Hamiltonian flow in M_μ induced by h_μ is given by the equations

$$\dot{r} = \frac{\partial h_\mu}{\partial p_r}, \qquad \dot{p}_r = -\frac{\partial h_\mu}{\partial r}.$$

Note that the Manev reduced potential V_μ is identical to the one corresponding to the Kepler problem with momentum equal to $\sqrt{l^2 - 2mB}$. In other words, the reduced Manev system with momentum $\mu = (0, 0, l)$, is identical to the reduced Kepler system with momentum $\mu' = (0, 0, \sqrt{l^2 - 2mB})$. Hence, up to this momentum shift, the reduced dynamics of both systems are identical. It is the geometrical phase that lifts the dynamics in M_μ to M that differentiates between the Kepler and the Manev systems.

We will focus on the bounded motions of the reduced system. These motions occur provided that the total momentum of the system l, satisfies

$$l > \sqrt{2mB}.$$

In such a case, the effective potential $V_\mu(r)$, looks like the one in Fig. 6.

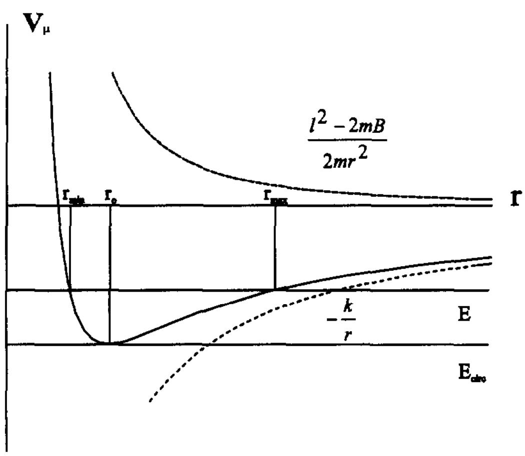

Fig. 6. One effective potential in the Manev problem.

The particular form of V_μ, for negative values E of the energy, forces the r variable to be bounded between the values $r_{\min}(E)$ and $r_{\max}(E)$ that are given as the solutions of the quadratic equation

$$E = -\frac{k}{r} + \frac{l^2 - 2mB}{2mr^2},$$

that is,

$$r_{\max}(E) = -\frac{1}{2E}\left(k + \sqrt{\frac{mk^2 + 2El^2 - 4BEm}{m}}\right) \tag{5.9}$$

$$r_{\min}(E) = -\frac{1}{2E}\left(k - \sqrt{\frac{mk^2 + 2El^2 - 4BEm}{m}}\right). \tag{5.10}$$

Also, $V_\mu(r)$ admits a unique minimum at the value $r_0 = (l^2 - 2mB)/mk$, for which

$$V_\mu(r_0) = -\frac{mk^2}{2(l^2 - 2mB)} := E_{\text{circ}}.$$

If the energy E of the system is such that $E = E_{\text{circ}}$, there is a circular orbit, which is a relative equilibrium with respect to the $SO(3)$ symmetry. Its stability can be studied using the energy–momentum method. If the energy is such that $0 > E > -mk^2/(2(l^2 - 2mB))$, the system describes an RPO which we prove by showing that the motion in M_μ is periodic using a method identical to the one followed in the case of the spherical pendulum (based on Proposition 5.6). The only difference is that in this case, r and p_r are related by

$$p_r = \pm\sqrt{2m\left(E + \frac{k}{r} - \frac{l^2 - 2mB}{2mr^2}\right)}.$$

These periodic orbits lift by Theorem 5.2 to RPOs in $T^*\mathbb{R}^3$. Since $\dim M_\mu = 2$, Theorem 5.4 guarantees that these orbits are $SO(2)$-stable.

Remark 5.7. *Notice that the treatment utilized above for the Manev potential can be adapted to any central potential; in other words, any RPO created by a central potential in three dimensions is going to be $SO(2)$-stable, provided that its corresponding momentum value is regular. This is so since, in this case, the dimension of the corresponding reduced space is always equal to two, the associated reduced periodic point is orbitally stable and therefore, by Theorem 5.4, the RPO is $SO(2)$-stable.*

Acknowledgements

We thank F. Fassò, J. Marsden, A. Weinstein, and the referees for critically listening to us and giving us advice on various aspects of this paper. The research of the first author was partially supported by a Fulbright/BCH Fellowship, the US Information Agency and a

Rotary Ambassadorial Scholarship. The research of the second author was partially supported by NSF Grant DMS-9802378, DOE Contract DE-FG03-95ER25245-A000, and the Swiss NSF.

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